United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Thursday 5 November 2020

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For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation.

There is also a version of this document available on the UKMT website which includes each of the questions alongside its solution:

www.ukmt.org.uk

1. **D** The value of $\frac{2020}{20\times 20} = \frac{101}{20} = \frac{50.5}{10} = 5.05$.

2. **C** When the product 1234×5678 is calculated, its units digit is the units digit of 4×8, so 2. This is the remainder when the product is divided by 5.

3. **A** The central cube contributes two faces to the surface area of the shape. The other four cubes contribute five faces each. This gives a surface area of $2 + 4 \times 5 = 22$ faces. The surface area is then 22 as each face has area 1 unit.

4. **B** As $p = 2$, $q = \frac{20}{2} = 10$. Then $20 \times r = 202$ so $r = \frac{202}{20} = 10.1$. Finally, $202 \times s = 2020$ so $s = \frac{2020}{202} = 10$. The value of $p + q + r + s = 2 + 10 + 10.1 + 10 = 32.1$.

5. **C** Consider each option to be in the form $1.11\ldots \times 10^k$. When numbers of this form are squared, they become $1.2\ldots \times 10^{2k}$. As the question asks for $\sqrt{123454321} = \sqrt{1.2\ldots \times 10^8}$, the value of k is 4. Thus the correct option is C, 11111.

6. **E** The number of students in the maths class must be a multiple of 2, 4 and 7. As there are known to be fewer than 30 students, there must be 28 students in the class, which is the only common multiple of 2, 4 and 7 less than 30. Therefore the number of students who play hockey is $\frac{28}{4} = 7$.

7. **D** As a proportion, $\frac{47}{225} \approx \frac{45}{225} = 20\,\%$. So the percentage reduction in reported teapot accidents is approximately $80\,\%$.

8. **D** Using the difference of two squares, $106^2 - 15^2 = (106 + 15)(106 - 15) = 121 \times 91$, which when expressed as the product of primes is $11 \times 11 \times 7 \times 13$. The largest of these primes is 13.

9. **B** Using distance = speed × time, distance = 180 km/h × 1 second, which written in metres and seconds is $\dfrac{180 \times 1000\,\text{m}}{60 \times 60\,\text{seconds}} \times 1\,\text{second} = 50\,\text{m}$.

10. **C** The first piece of information allows us to place P and Q in four possible pairings as shown. In each case, the second piece of information tells us that the position of Roman is uniquely determined and in each case this is next to Pat. As Sam is not at the end of the line, the third piece of information shows us that Sam's position is also uniquely determined and is next to Qasim. The positions which remain for Tara and Uma, in each of the four cases, have two people between them.

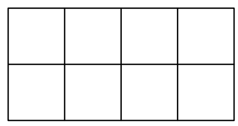
P R _ S Q _
or _ P R _ S Q
or Q S _ R P _
or _ Q S _ R P

11. **E** The perimeter of the octagon is made from four long sides, two medium-length sides and two short sides. The long sides are given to be of length 1. The medium-length sides have length $\frac{1}{\sqrt{2}}$, using Pythagoras' Theorem on the right-angled triangle which was removed from the original square. Therefore the length of each short side is $1 - \frac{1}{\sqrt{2}}$. In total the perimeter has length $4 \times 1 + 2 \times \frac{1}{\sqrt{2}} + 2 \times (1 - \frac{1}{\sqrt{2}}) = 6$.

12. **B** In pounds, let the price of a jacket $= j$, the price of a pair of trousers $= t$ and the price of a waistcoat $= w$. We are given that $2j + 3t = 380$ and that $t = 2w$. We want to know the value of $j + t + w$. Rewriting the first equation as $2j + 2t + t = 380$ and substituting $2w$ instead of the final t gives $2j + 2t + 2w = 380$. Dividing by 2 then gives us $j + t + w = 190$ and so the cost of the three-piece suit is £190.

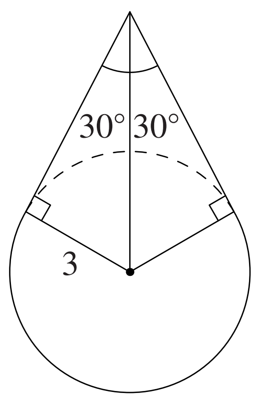
13. **D** When 16! is written out as the product of its primes, $16! = (2^4) \times (3 \times 5) \times (2 \times 7) \times (13) \times (2^2 \times 3) \times (11) \times (2 \times 5) \times (3^2) \times (2^3) \times (7) \times (2 \times 3) \times (5) \times (2^2) \times (3) \times (2)$ in which the power of 2 is $2^{4+1+2+1+3+1+2+1} = 2^{15}$. In order for $16! \div 2^k$ to be an odd integer, 2^k must exactly equal 2^{15}, so $k = 15$.

14. **B** One counting method is to consider first the position of the two red disks working systematically from left to right. If there is a red disk in either of the two positions in the first column, then there are five possible positions for the second red disk.



If there are no red disks in the first column but one in either of the two positions in the second column then there are three possible positions for the second red disk. If there are no red disks in either of the first two columns but there is one in either of the two positions in the third column then there is exactly one position for the second red disk. This gives a total of $2 \times 5 + 2 \times 3 + 2 \times 1 = 18$ different looking positions for the two red disks. The single yellow disk can then be placed in any of the remaining six squares. The last five positions are filled with the five blue disks. This gives $18 \times 6 \times 1 = 108$ possibilities.

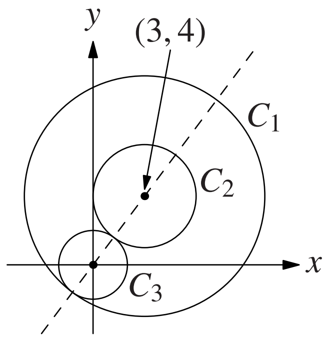
15. **A** Drawing three extra lines on the diagram, vertically from the centre of the circle to the top vertex and also from the centre to the points where the tangents touch the circle, allows us to see the shaded area as two triangles and a sector outside the triangles. The tangents are perpendicular to the radii and each triangle has the vertical line as one of its sides, so using 'two sides and a right-angle' the two triangles are congruent. Therefore the triangles have angles of 30°, 60° and 90°. As we are given that the radius is 3, the length of the tangents is $3\sqrt{3}$ and so the area of each triangle is $\frac{1}{2} \times 3 \times 3\sqrt{3} = \frac{9\sqrt{3}}{2}$. The angle between the two radii is 120° so the sector is $\frac{2}{3}$ of the area of the circle of radius 3 which is $\frac{2}{3} \times \pi \times 3^2 = 6\pi$. The total area is then $6\pi + 2 \times \frac{9\sqrt{3}}{2} = 6\pi + 9\sqrt{3}$.



16. E Rearranging $y^2 - 2y = x^2 + 2x$ gives $y^2 - x^2 = 2x + 2y$ then factorising on each side gives $(y+x)(y-x) = 2(x+y)$ and so $(y-x-2)(y+x) = 0$. Therefore either $y = x+2$ or $y = -x$. A sketch of those two straight lines is shown in option E.

17. A As m is a positive integer, in this equation $3m$ could equal 3, 6, 9, 12 or 15 leaving $\dfrac{3}{n+\frac{1}{p}} = 14$, 11, 8, 5 or 2. However $n, p > 1 \implies n + \frac{1}{p} > 1 \implies \dfrac{3}{n+\frac{1}{p}} < 3$. Therefore the only possibility here is that $\dfrac{3}{n+\frac{1}{p}} = 2$. So $\frac{3}{2} = n + \frac{1}{p}$ which is possible only when $n = 1$ and $p = 2$.

18. E The centres of all three circles along with both points where the circles touch each other lie on the same straight line which passes through $(0,0)$ and $(3,4)$. Using Pythagoras' theorem the distance between $(0,0)$ and $(3,4)$ is 5 units. Circle C_1 has radius $5+2 = 7$ and circle C_2 has radius $5-2 = 3$ so the sum of the radii of those two circles is $7+3 = 10$.



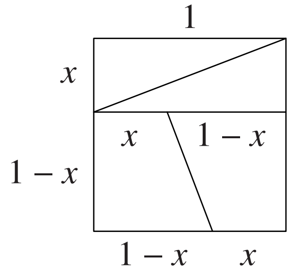
19. A Combining $p - q = r$ and $r - s = t$ gives $p - q = s + t$ which rearranges to $t = p - (q+s)$. The maximum value of t can be found by maximising p and minimising the sum of q and s, so $t_{\text{max}} = 9 - (1+2) = 6$. The minimum value of t is 1, for example $t_{\text{min}} = 9 - (2+6) = 1$. All the values of t from 1 to 6 are possible in a similar way. Therefore t can have six different values.

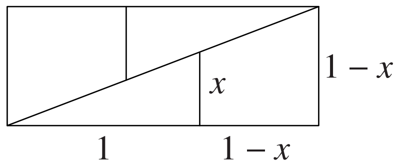
20. E Rewriting both sides of the first equation using a base of $\sqrt{2}$ gives $\left(\sqrt{2}^4\right)^y = \frac{1}{\sqrt{2}^6\sqrt{2}^{x+2}}$ so $\sqrt{2}^{4y} = \sqrt{2}^{-(x+8)}$ and therefore $4y = -(x+8)$. Rewriting the second equation with a base of $\sqrt{3}$ gives $\left(\sqrt{3}^4\right)^x \times \left(\sqrt{3}^2\right)^y = \sqrt{3}^3$. Again, employing rules of indices leads to $4x + 2y = 3$. Solving simultaneously the pair of linear equations shows that $x = 2$ and $y = -\frac{5}{2}$ and therefore $x + y = -\frac{1}{2}$. The value of $5^{x+y} = 5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$.

21. C The expression $(10^{2020} + 2020)^2$ can be written as $(10^{2020} + 2020)(10^{2020} + 2020)$ and expanded to give $10^{4040} + 4040 \times 10^{2020} + 4080400$. The sum of the digits of the 4041-digit number is the sum of the non-zero digits. Since there is no overlap of the positions of the non-zero digits of the three parts of the expanded expression, this sum is $1+4+4+4+8+4 = 25$.

22. A The perimeter of the square is 4 therefore each side is of length 1 and the area of the square is 1. The square must be cut so that the shortest side of each triangle matches the shorter of the two parallel sides of the trapezium when the pieces are rearranged.

Let each of these sides be of length x and then the remaining perpendicular sides of each trapezium have length $1 - x$ as indicated on the first diagram. The perimeter of the rectangle has length $2 \times 1 + 4(1 - x) = 6 - 4x$.



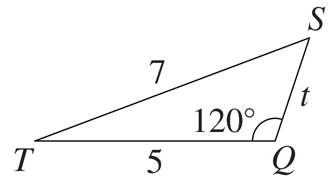


In order to find the value of x, we can consider the area of the rectangle so $(1 + 1 - x)(1 - x) = 1$. This rearranges to give $x^2 - 3x + 1 = 0$ and therefore, of the two possible solutions, $x = \frac{3-\sqrt{5}}{2}$ as $x < 1$. The perimeter is $6 - 4(\frac{3-\sqrt{5}}{2}) = 2\sqrt{5}$.

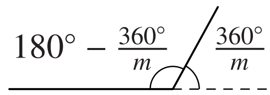
23. D Letting $y = 3$ and rearranging gives $\dfrac{f(x)}{f(3)} = \dfrac{x^3}{3^3}$. Therefore $\dfrac{f(20) - f(2)}{f(3)} = \dfrac{f(20)}{f(3)} - \dfrac{f(2)}{f(3)} = \dfrac{20^3}{3^3} - \dfrac{2^3}{3^3} = \dfrac{8000 - 8}{27} = \dfrac{7992}{27} = 296$.

24. B Triangle PTS can be shown to be isosceles as follows. Let $\angle STQ = 2x^\circ$. Therefore $\angle RPT = 2x^\circ$ as PR and ST are parallel. We can then deduce that $\angle SPT = x^\circ$ as PS bisects $\angle RPQ$. Considering angles at point T, $\angle STP = 180^\circ - 2x^\circ$. Finally, considering angles inside triangle PTS, $\angle PST = 180^\circ - x^\circ - (180^\circ - 2x^\circ) = x^\circ = \angle SPT$. Therefore the length of ST = the length of $PT = \frac{12}{2} + 1 = 7$.

Now considering triangle STQ, we have two known lengths and an angle so we can find the length of SQ using the cosine rule. Let the length of $SQ = t$. So $7^2 = 5^2 + t^2 - 2 \times 5 \times t \times \cos 120^\circ$. As $\cos 120^\circ = -\frac{1}{2}$, this quadratic simplifies to $t^2 + 5t - 24 = 0$ which factorises to $(t + 8)(t - 3) = 0$. As $t > 0$, $t = 3$.



25. C The exterior angle of the regular m-gon is $\frac{360^\circ}{m}$ so the interior angle is $180^\circ - \frac{360^\circ}{m}$. Similarly for the n-gon and p-gon.



Possible interior angles, in order of size, are then 60°, 90°, 108°, 120°, $(180^\circ - \frac{360^\circ}{7})$, 135°, 140°, 144°, ... In order to maximise p, the interior angles of the m-gon and n-gon must have a sum which exceeds 180° by as little as possible. This excess gives the exterior angle of the p-gon, so must fit into 360° an exact number of times. As a starting point, note that when $(m, n) = (4, 4)$ or $(m, n) = (6, 3)$ the interior angles of the m-gon and n-gon sum to exactly 180°. By increasing exactly one of m or n in these pairs, possible candidates for pairings which will give us a sum of interior angles in excess of 180° by a suitable amount are then $(5, 4)$, $(4, 5)$, $(6, 4)$, and $(7, 3)$. The value of p is then found from $p = \dfrac{360^\circ}{[180^\circ - \frac{360^\circ}{m}] + [180^\circ - \frac{360^\circ}{n}] - 180^\circ}$. Without loss, we can assume $m \geq n$ and therefore discount $(m, n) = (4, 5)$. When $(m, n) = (5, 4)$, $p = \frac{360^\circ}{[180^\circ - \frac{360^\circ}{5}] + [180^\circ - \frac{360^\circ}{4}] - 180^\circ} = \frac{360^\circ}{18^\circ} = 20$. As this is an integer, we don't need to try $(m, n) = (6, 4)$ as that could only give a smaller value of p. When $(m, n) = (7, 3)$, $p = \frac{360^\circ}{[180^\circ - \frac{360^\circ}{7}] + [180^\circ - \frac{360^\circ}{3}] - 180^\circ} = \frac{360^\circ}{60^\circ - \frac{360^\circ}{7}} = \frac{360^\circ}{\frac{60^\circ}{7}} = 42$. As increasing m or n or both any further could only give a smaller value of p, no further pairs need to be tested. Hence the largest value of p is 42.